বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - VI

Subject: MATHEMATICS

Paper: CC - 14 (Ring Theory and Linear Algebra II – Theory)

Full Marks: 60 (Theory)

Time: 3 Hours (Theory)

Candiates are required to give their answer in their own words as far as practicable.

Questions are of equal value.

Answer any **one question** from the following:

1. (a) Sate Eisenstein criterion. Prove that this is a sufficient criterion for irreducibility of a polynomial over $\mathbb{Z}$ (the set of all integers).

   (b) Check whether the polynomial $x^4 - 5x^2 + x + 1$ is irreducible or not over $\mathbb{Z}$.

   (c) In an integral domain, prove that any prime element is irreducible. Show that the converse of this result is not true in an integral domain.

2. (a) Consider the integral domain $\mathbb{Z}\left[\sqrt{-3}\right]$. Then Show that

   (i) 1 and –1 are the only units in this integral domain.

   (ii) $1+\sqrt{-3}$, 2 are irreducible elements in this integral domain.

(iii) But none of $1+\sqrt{-3}$ and 2 is prime there.

(b) Prove that the ring of Gaussian integers is a Euclidean domain.

3. (a) Give example of an integral domain which is not a factorization domain with justification.

(b) If K is a field then prove that $K[x]$ is a Euclidean domain.

(c) Let R be a commutative ring with identity such that $R[x]$ is a PID (Principal Ideal Domain). Then prove that R is a field.

(d) Check the irreducibility of a cyclotomic polynomial over $\mathbb{Z}$.

(e) Show that $R[x]/(x) \cong R$ where R is a commutative ring with identity.

4. (a) Define diagonalizability of a linear operator on a finite dimensional vector space. Let T be a linear operator on a vector space V over F such that $T^2 = T$. Prove that T is diagonalizable.

(b) Find all possible invariant factors and hence write down all possible Jordan canonical forms of a matrix over $\mathbb{R}$ (the field of all real numbers) whose characteristic polynomial is

$$(x-4)^4(x-1)(x-5)^5$$

and the minimal polynomial is

$$(x-4)^2(x-1)(x-5)^2.$$

5. (a) Let V be the vector space of all polynomial functions p from $\mathbb{R}$ to $\mathbb{R}$ which have degree two or less: $p(x) = c_0 + c_1x + c_2x^2$. Let us define the following three linear functionals on V by

$$f_1(p) = \int_0^1 p(x)dx \text{ for all } p \in V,$$

$$f_2(p) = f_0^2 p(x)dx \text{ for all } p \in V,$$

$$f_3(p) = f_0^{-1} p(x)dx \text{ for all } p \in V.$$

Show that $\{f_1, f_2, f_3\}$ forms a basis for V^* by exhibiting the basis for *V* of which it is the dual.

(b) Define transpose of a linear transformation. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (0, x, y)$. Find T^t (transpose of *T*).

(c) Let *A* be an $m \times n$ matrix with real entries. Prove that each entry of *A* is zero if and only if trace $(A^t A) = 0$.

(d) Let n be a positive integer and let *V* be the vector space of all polynomial functions over the field of real numbers which have degree at most *n*, i.e., the functions of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n.$$

Let *D* be the differentiation operator on *V*. Find a basis for the null space of the transpose operator D^t.

6. (a) Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$ equipped with the standard inner product. Applying Gram-Schmidt orthogonalization process, orthonormalize the following set of vectors:

{(1, 0, 1), (1, 0, –1), (0, 3, 4)}.

(b) Let *V* be a finite dimensional inner product space and *f* be a linear functional on *V*. Then prove that there exists a unique vector β in *V* such that $f(\alpha) = \langle \alpha, \beta \rangle$ for all $\alpha \in V$.

(c) Let *T* be a linear operator on a finite dimensional inner product space *V*. Show that there exists a unique linear operator T^* on *V* such that

$\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in V$.

(d) Define self-adjoint operator and normal operator on a finite dimensional inner product space. Let *V* be a finite dimensional inner product space and *T* be a normal operator on *V*. Then prove that α is a characteristic vector for *T* with characteristic value *c* if and only if α is a characteristic vector for T^* with characteristic value $\overline{c}$.

7. (a) Let T be the linear operator on $\mathbb{R}^2$ over $\mathbb{R}$ which is represented in the standard ordered basis by the following matrix:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Find all the invariant subspaces of $\mathbb{R}^2$ which are invariant under T.

(b) Let T be the linear operator on a finite dimensional vector space V over the field F. Define minimal polynomial of T. Prove that T is diagonalizable if and only if the minimal polynomial p of T has the form $p(x)=\prod_{i=1}^{k}(x-c_t)$ where $c_1, c_2, ..., c_k$ are distinct elements of F.

(c) Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the following matrix:

$$\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$$

Exhibit a basis for $\mathbb{R}^3$ with respect to which the matrix representation of T is diagonal.

8. (a) If $f(x)$ is a polynomial in $F[x]$ of degree 2 or 3, then show that $f(x)$ is reducible over the field F if and only if it has a zero in F.

Let $f(x)=x^4-2x^3+x+1$. Show that $f(x)$ is irreducible over $\mathbb{Q}[x]$.

(b) Define an irreducible element and a prime element in an integral domain D. Show that every irreducible element is a prime element in a unique factorization domain.

(c) Show that 2 is an irreducible element in the domain $D=\mathbb{Z}\left[\sqrt{-6}\right]$. Using the equality $2.5=\left(2+\sqrt{-6}\right)\left(2-\sqrt{-6}\right)$, establish that 2 is not a prime element in D.

9. (a) Define a unique factorization domain. Give an example of it with explanation.

(b) Let D be a PID (principal ideal domain) and $< p >$ be a nonzero ideal in D. Show that $< p >$ is a maximal ideal if and only if p is irreducible.

(c) (i) Define a Euclidean domain. Prove that every Euclidean domain is a PID.

(ii) Find a *gcd* (d) of the elements $a = 7+4i$ and $b = 4+3i$ in $\mathbb{Z}[i]$ with a Euclidean valuation v defined by $v(m+in) = m^2 + n^2$ for $m+in \in \mathbb{Z}[i]$. Also express d as $d = au + bv$ for some $u, v \in \mathbb{Z}[i]$.

10. (a) State and prove Cayley-Hamilton theorem. Use the theorem to find A^{100},

where = $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(b) Find an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

11. (a) Define the dual space of a vector space V.

If V is a vector space of dimension n over a field F, then the dimension of its dual space is also n.

(b) Let a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$T(x, y, z) = (2x + y - 2z, 2x + 3y - 4z, x + y - z)$. Find all eigen values of T and find a basis of each eigen space.

(c) Reduce the following quadratic form into Canonical form and examine whether it is positive definite or not: $6x^2 + y^2 + 18z^2 - 4yz - 12zx$.

12. (a) Find the value of k so that the following expression forms an inner product:

$(\vec{a}, \vec{b}) = a_1b_1 - 3a_1b_2 - 3a_2b_1 + ka_2b_2$, where $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$.

Use the Gram-Schmidt orthogonalisation process to find the orthonormal basis of $\mathbb{R}^3$ generated by the set of vectors {(1, –1, 1), (2, 0, 1), (0, 1, 1)}.

(b) If $\{\beta_1, \beta_2, ..., \beta_r\}$ is an orthonormal set of vectors in a Euclidean space V, then for any vector α in V, show that $\|\alpha\|^2 \geq c_1^2 + c_2^2 + ... + c_r^2$, where c_i is the scalar component of α along β_i, $i = 1, 2, ..., r$. When does the equality occur?

(c) Find the orthogonal complement of the subspace P, generated by the vectors (1, 1, 0) and (0, 1, 1) in $\mathbb{R}^3$.