বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2020

(Under CBCS Pattern)

Semester - VI

Subject: MATHEMATICS

Paper: CC - 13 (Metric Spaces and Complex Analysis – Theory)

Full Marks: 60 (Theory)

Time: 3 Hours (Theory)

Candiates are required to give their answer in their own words as far as practicable.

Questions are of equal value.

Answer any **one question** from the following:

Metric Spaces and Complex Analysis (Theory)

1. (a) Define Cauchy sequence in a metric space. Give an example of a Cauchy sequence in a specified metric space.

   (b) Prove that every convergent sequence is a Cauchy sequence. Show by an example that a Cauchy sequence may not converge.

   (c) Prove that a Cauchy sequence $\{x_n\}$ in a metric space (X, d) converges if and only if it has a convergent subsequence $\{x_{n_k}\}$.

(d) If (X, d) be a metric space with the metric defined by $d(x, y)=\begin{cases}0 \text{ if } x=y\\1 \text{ if } x\neq y\end{cases}$, then show that (X, d) is complete.

(e) Show that the space $X=(0,\ 1]$ with usual metric $d(x, y)=|x-y|,\ \forall x, y\in X$, is not complete.

2. (a) Let (X, d) and (Y, d') be two metric spaces. Prove that a function $f:(X, d)\to(Y, d')$ is continuous at a point $x\in X$, if and only if for all sequences $\{x_n\}$ of elements of X converging to the point x in (X, d), the sequences $\{f(x_n)\}$ of elements of Y converge to $f(x)$ in (Y, d').

(b) If A and B are two non-empty disjoint closed sets in a metric space (X, d), then show that there exists a continuous function $f:(X, d)\to R$ such that $f(x)=\begin{cases}1 \text{ if } x\in A\\0 \text{ if } x\in B\end{cases}$.

(c) State Heine-Borel Property. Show that the real line is not compact.

(d) Define finite intersection property (F.I.P). Does the collection $A=\{(-n, n): n\in N\}$ of open intervals satisfy finite intersection property ?

3. (a) Let (X, d) be a metric space, and $\{x_n\}$ and $\{y_n\}$ are convergent sequences in this metric space converging to x and y, respectively. Then prove that the sequence $\{d(x_n, y_n)\}$ is convergent in the real line and converges to $d(x, y)$.

(b) Define complete metric space. Prove that $(\mathbb{C}, d)$ is a complete metric space where $\mathbb{C}$ is the set of complex numbers and d is the metric defined on $\mathbb{C}$ by

$$d(x, y)=|x-y|,\ \forall\ x, y\in\mathbb{C}.$$

(c) Let $C[a, b]$ be the space of continuous functions over the bounded closed interval $[a, b]$ equipped with the metric d given by

$$d(f, g)=\sup\{|f(x)-g(x)|: x\in[a, b]\},\ \forall\ f, g\in C[a, b]$$

Then prove that a sequence of functions $\{f_n\}$ converges to a function *f* in the metric space $(C[a, b], d)$ if and only if the sequence of functions $\{f_n\}$ converges uniformly to *f* in $[a, b]$.

4. (a) State and prove the Cantor intersection theorem on metric space.

   (b) Define the continuity of a function on a metric space. If (X, d_x) and (Y, d_y) be two metric spaces, then prove that the function $f(X, d_x) \to (Y, d_y)$ is continuous at $x \in X$ if and only if every sequence $\{x_n\}$ converges to *x* in (X, d_x), the sequence $\{f(x_n)\}$ converges to $f(x)$ in (Y, d_y).

   (c) Let *S* be a non-empty subset of a metric space (X, d_x), then prove that the function $f : X \to \mathbb{R}$ given by $f(x) = d(x, S),\ \forall x \in X$ is continuous, by taking usual metric $d_{\mathbb{R}}$ on $\mathbb{R}$, i.e., $d_{\mathbb{R}}(a, b) = |a - b|,\ \forall a, b \in \mathbb{R}$.

5. (a) Define uniform continuity of a function on a metric space. Prove that the composition of two uniformly continuous functions on a metric space is also uniformly continuous on the same metric space.

   (b) What do you mean by separated sets on a metric space? Prove that in a metric space two open sets are separated if and only if they are disjoint.

   (c) Prove that in a metric space, the continuous image of a connected set is connected.

6. (a) What do you mean by compact metric space and compact set? Prove that in a metric space every compact set is a closed set.

   (b) State and prove Heine Borel Theorem.

   (c) Give example of a metric space where a closed and bounded set may not be compact.

7. (a) Prove that convergence sequence of complex numbers is bounded. Is the converse true ? Justify.

(b) Check whether the following sequences are convergent :

(i) $\left\{\frac{2^n}{n!}+i\frac{n}{2^n}\right\}$

(ii) $\left\{n(1+i)^n\right\}$.

(c) Suppose $\lim_{z\to z_0} f(z)=l_1$ and $\lim_{z\to z_0} g(z)=l_2$ then prove that

$$\lim_{z\to z_0}\left(\frac{f(z)}{g(z)}\right)=\frac{l_1}{l_2}(l_2\neq 0)$$

(d) Find the following limits, if exists:

(i) $\lim_{z\to 0}\frac{lm(z)}{z+|z|^2+2}$, and

(ii) $\lim_{z\to 0}\left[\frac{1}{1-e^{\frac{1}{x}}}+iy^2\right]$

(e) Test the continuity of the following functions:

(i) $f(z)=\begin{cases} z\frac{\mathrm{Re}(z)}{|z|}, & z\neq 0 \\ 0, & z=0 \end{cases}$

(ii) $f(z)=\begin{cases} \frac{(\mathrm{Re}(z))^3(1+i)-(lm(z))^3(1-i)}{|z|^2} & z\neq 0 \\ 0, & z=0 \end{cases}$, and

(iii) $f(z)=\begin{cases} \frac{(\mathrm{Re}(z))^3}{|z|^2} & z\neq 0 \\ 0, & z=0 \end{cases}$

8. (a) Write sufficient conditions for a function $f(z)$ to be differentiable.

(b) Define analytic function. Give an example of function which satisfies Cauchy Riemann's Equations but not differentiable.

(c) Check which of the following functions are analytic functions?

(i) $f(z) = \bar{z} + |z|^2$, and

(ii) $f(z) = e^{-|z|^4} + z + 6$

(d) If $f(z)$ is analytic function of *z*, then prove that

(i) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$, and

(ii) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$

(e) Find the following integrals:

(i) $\int_L \frac{e^z}{(z-i)^3(z-1)^2}dz,\ L: \left|z - \frac{i}{2}\right| = 1$, and

(ii) $\int_L \frac{e^z}{z^2(z-3)^3}dz,\ L$ is the square with vertices (4, –4), (4, 4), (–4, 4), (–4, 4).

9. (a) Evaluate $\int_\Gamma \bar{z}dz$, where Γ is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$.

(b) Find an upper bound of $\int_C \frac{1}{(z^4+1)^2}dz$, where *C* is the upper half circle $|z| = a,\ a > 1$, traversed once in the counter clock wise direction.

(c) Define Cauchy integral formula and hence find $\int_C \frac{z}{(9-z^2)(z+i)}dz$, *C* is the circle $|z| = 2$.

(d) Find the following integrals:

(i) $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$, where C is the circle $|z-i|=3$, and

(ii) $\int_C \frac{1}{z^2+2z+5}dz$, where C be a circle centred at 4 + i with radius 1.

(e) Find the Taylor series expansion of the following functions:

(i) $f(z)=\frac{z}{z^4+9}$ about $z=0$,

(ii) $f(z)=\log(1+z)$ about z = 0,and

(iii) $f(z)=\sin(z)$ about $z=\frac{\pi}{4}$.

10. (a) State and prove Liouville's theorem.

(b) Is $\sin(z)$ a bounded function in complex plane? Justify.

(c) Check the convergence of the following series:

(i) $\sum_{n=1}^{\infty} e^{\frac{in\pi}{6}}$,

(ii) $\sum_{n=1}^{\infty} \frac{1}{(1+i)^n}$, and

(iii) $\sum_{n=1}^{\infty} \left(\frac{n+i}{n^3}\right)$.

(d) Expand $f(z)=\frac{15z+1}{(z-4)(z-6)}$ about $z_0=3$. Suggest the region of validity of this representation.

(e) Obtain all possible Laurent series expansion of the following functions:

(i) $f(z)=\frac{\sin(z^2)}{z^5}$ about $z_0=0$,and

(ii) $f(z)=\dfrac{1}{z^2(z-i)}$ about $z_0=i$.

11. (a) Define limit of a complex valued function at a point.

(b) If a complex valued function, $f(z)=u(z)+iv(z)$, $z=x+iy$, be defined on $D\subseteq\mathbb{C}$, except possibly at $z_0=x_0+iy_0$, where, $u(z)=u(x,y)$ and $v(z)=v(x,y)$. Then prove that the following limit

$$\lim_{x\to z_0} f(z)=w_0=l_1+il_2\text{, where, } l_1, l_2\in\mathbb{R}$$

holds, if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y)=l_1 \quad\text{and}\quad \lim_{(x,y)\to(x_0,y_0)} v(x,y)=l_2 .$$

(c) If two complex valued functions $f(z)$ and $g(z)$, $z=x+iy$, be defined on, $D\subseteq\mathbb{C}$ such that

$$\lim_{z\to z_0} f(z)=l_1 \text{ and } \lim_{z\to z_0} g(z)=l_2 .$$

Then prove that, $\lim\limits_{z\to z_0}\dfrac{f(z)}{g(z)}=\dfrac{l_1}{l_2}$, where, $l_2\neq 0$.

(d) Test the existence of the limit, $\lim\limits_{z\to 0} f(z)$, where, $f(z)=\dfrac{\overline{z}}{z}$, $z=x+iy$.

12. (a) Define uniform continuity of a complex valued function.

(b) If a complex valued function *f* is continuous on a compact set, *D*, then prove that it is uniformly continuous there. Is the converse true?

(c) Prove that the composite of two complex valued continuous functions is continuous.

(d) Let a function $f(z)=u(r,\theta)+iv(r,\theta)$ be analytic in a domain *D* which does not include the origin. Then prove that $u(r,\theta)$ satisfies the following relation

$$r^2u_{rr}(r,\theta)+ru_r(r,\theta)+u_{\theta\theta}(r,\theta)=0$$

throughout the domain *D*.